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Extension of a combined analytical/numerical initial value problem solver for unsteady periodic flow

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SUMMARY

Here we describe analytical and numerical modifications that extend the *D*ifferential *R*educed *E*jector/mixer *A*nalysis (DREA), a combined analytical/numerical, multiple species ejector/mixing code developed for preliminary design applications, to apply to periodic unsteady flow. An unsteady periodic flow modelling capability opens a range of pertinent simulation problems including pulse detonation engines (PDE), internal combustion engine ICE applications, mixing enhancement and more fundamental fluid dynamic unsteadiness, e.g. fan instability/vortex shedding problems. Although mapping between steady and periodic forms for a scalar equation is a classical problem in applied mathematics, we will show that extension to systems of equations and, moreover, problems with complex initial conditions are more challenging. Additionally, the inherent large gradient initial condition singularities that are characteristic of mixing flows and that have greatly influenced the DREA code formulation, place considerable limitations on the use of numerical solution methods. Fortunately, using the combined analytical–numerical form of the DREA formulation, a successful formulation is developed and described. Comparison of this method with experimental measurements for jet flows with excitation shows reasonable agreement with the simulation. Other flow fields are presented to demonstrate the capabilities of the model. As such, we demonstrate that unsteady periodic effects can be included within the simple, efficient, coarse grid DREA implementation that has been the original intent of the DREA development effort, namely, to provide a viable tool where more complex and expensive models are inappropriate.

KEY WORDS: periodic flow; combined analytical/numerical method; aerodynamic mixing; ejector nozzle

INTRODUCTION

Ejector–mixer nozzle systems provide an important acoustic and thermal treatment technology for high-speed civilian and military concepts, as well as, for currently deployed civilian (Hushkit) and military turbofan propulsion systems [1]. Design of these propulsion systems

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with an appropriate assessment of the scope of the design space at an adequate level of model fidelity, remains challenging. To begin to answer this need, a computer program *D*ifferential *R*educed *E*jector/mixer *A*nalysis (DREA) has been developed with the ability to run sufficiently fast so that it may be used either as a subroutine or called by an design optimization routine [1–5].

DREA is an implementation based on a combined perturbation/numerical modelling methodology that provides a rigorously derived family of solutions that require minimal empirical input. The base mathematical model is computationally more efficient than classical boundary layer but provides important two-dimensional information not available using quasi-1-d approaches [6]. To resolve singular behaviour, the model utilizes classical analytical solution techniques. Hence, analytical methods have been combined with efficient numerical methods to yield a hybrid fluid flow model.

Preliminary design models for ejector–mixer nozzles have typically involved control volume based approaches [7–11]. Though simple and robust, control volume based models cannot make any direct prediction about the streamwise length required to achieve a desired level of mixing. An estimate of length or equivalently mixing rate is essential for aerospace design applications since length required for mixing translates directly to weight, a critical flight design constraint. Early models employed boundary layer or 2-d, inviscid (method of characteristics) formulations to provide this type of information [12]. These models require that the primary stream be supersonic. Turbulent boundary layer formulations include the studies by Hedges and Hill [13, 14]. These method-of-characteristics and boundary layer models provide considerably more information than their control volume based, 1-d counterparts, though again, at greater computational cost. Further, boundary layer methods require the external imposition of a pressure field, predicted either using free stream information or through a global mass conservation constraint. This approximation may be poor for flows where the inlet static pressure of the streams is significantly different.

Here we describe analytical methods and codes modification that extend the DREA implementation to apply to periodic unsteady flow. This paper describes a continuation of the process to extend the capabilities of the DREA code to a range of physically more challenging problems. Major previous developmental efforts have provided a capability for multiple stream mix and multiple species reactive/combusting flow models [3, 5, 15]. A logical extension to the current spatially complex but temporally steady model, is an unsteady, periodic flow capability. An unsteady periodic flow modelling capability opens a range of interesting and pertinent simulation problems. Examples include, pulse detonation engine (PDE) (Figure 1) and internal combustion engine ICE applications.

Fluid dynamic unsteadiness, e.g. fan instability/vortex shedding and turbo-machinery flow interaction (Figure 2) might also be modelled using this technique. These applications (especially coupled with a combustion modelling capability) are particularly relevant for military, space and industrial applications.

Reduction of linear (or linearizable) periodic governing equations to a steady form is a classical problem in applied mathematics with close ties to method of normal modes [6]. Of course, the converse, i.e. extension of steady relationships to periodic structure is equally well founded. We will discuss these classical forms as motivation to our problem. However, we will see that, although single variable relationships are easily extended to periodic flow, extension of systems of equations and (moreover) problems with complex initial conditions are challenging to extend. The inherent large gradient initial condition singularities (which have

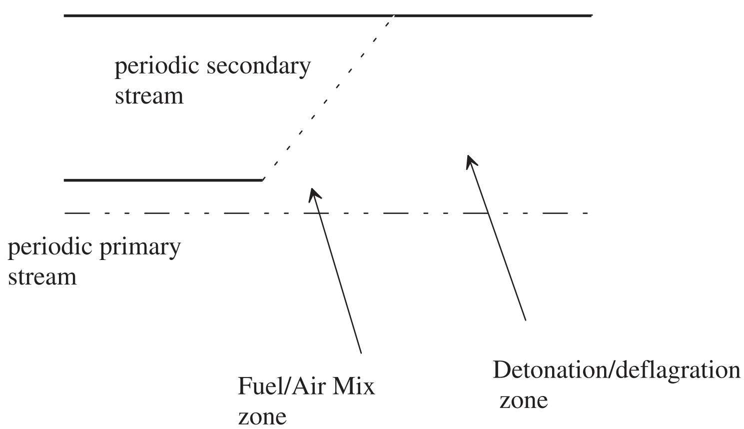

Figure 1. Pulse detonation engine wave cycle flow: a flow characterized by periodic, unsteady reactive processes.

greatly influenced all portions of the DREA code development) once again will place considerable limitations on the use numerical solution methods. Fortunately, however, the unique combined analytical–numerical form of the DREA formulation provides considerable flexibility in developing solution methods. Successful resolution of multiple stream periodic flow in the presence of large initial gradients will be directly dependent combined analytical/numerical solution methods [1].

Extension of the DREA mixing model to accept periodic flow has been facilitated by the general/canonical nature of the basic DREA formulation. References [1–4] provide a detailed discussion concerning the derivation of the DREA governing equations by performing an order of magnitude/perturbation expansion of more complete conservation relationships. A multiple species (currently three-phases, fuel, oxidizer and product) version of the DREA implementation [5] uses transport equations for momentum flux, global mass flux, total enthalpy flux and species mass fluxes, $i=1$–3, (note an inert phase is available through the mass constraint $\sum_{i=1}^{4} Y_i = 1$ but does not require a transport relationship):

$$\phi(x,y) = \begin{pmatrix} \rho u^2 + p \\ \rho u H \\ \rho u \\ \rho u Y_i \end{pmatrix} \tag{1}$$

with the basic transport equation (notice that it is linear in terms of the fluxes shown in Equation (1)):

$$\frac{\partial \phi}{\partial x} = a(x)\frac{\partial^2 \phi}{\partial y^2} + S_0 \phi \tag{2}$$

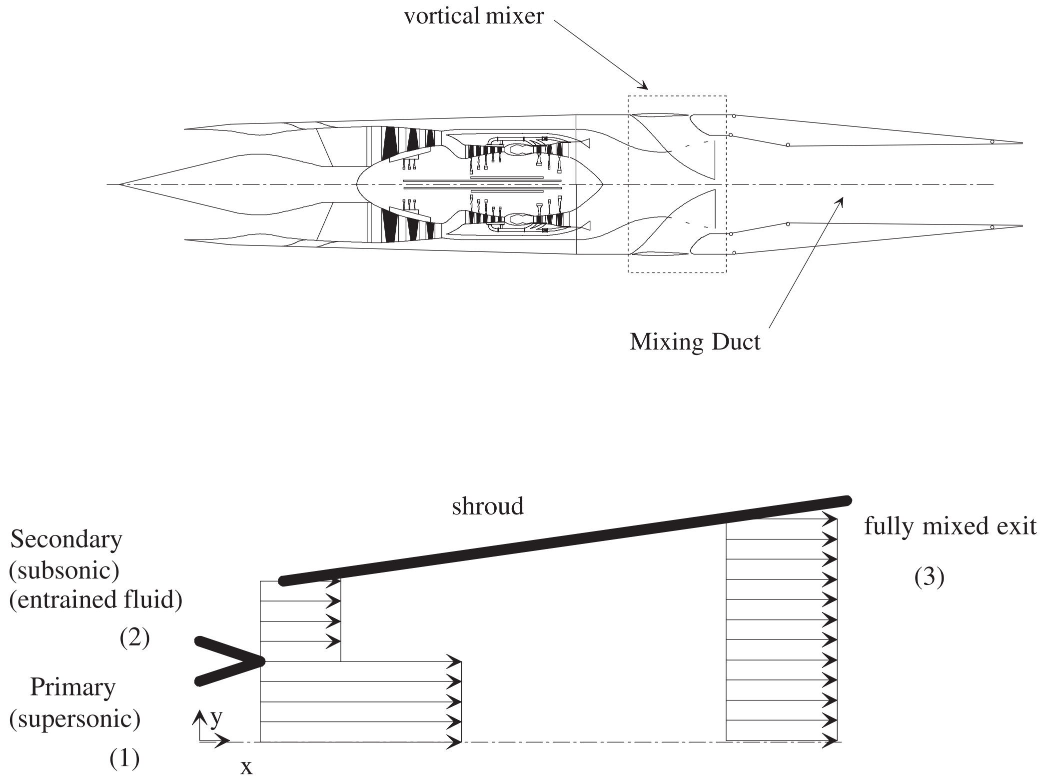

Figure 2. An ejector–mixer nozzle/turbofan deployment. Vortex shedding (splitter plate) and fan stream initial conditions are characterized by periodic unsteadiness.

the basic problem:

$$\frac{\partial\phi(x,0)}{\partial y}=\frac{\partial\phi(x,1)}{\partial y}=0 \tag{3}$$

with the initial condition:

$$\phi(0,y)=\begin{pmatrix}\phi_{10} 0\leqslant y\leqslant h_{\mathrm{s}}\\ \phi_{20} h_{\mathrm{s}}<y\leqslant 1\end{pmatrix} \tag{4}$$

for the two-stream case. Formulation and the approximate linearization of the governing transport equations for the DREA model in terms of flux functions is one of its unique features, and is described in detail in References [1, 4]. The associated turbulence model is described in Reference [2]. Detailed information concerning the combustion formulation is provided in Reference [5], including special linearization and solution procedures required for modelling combustion in the presence of large initial condition gradients.

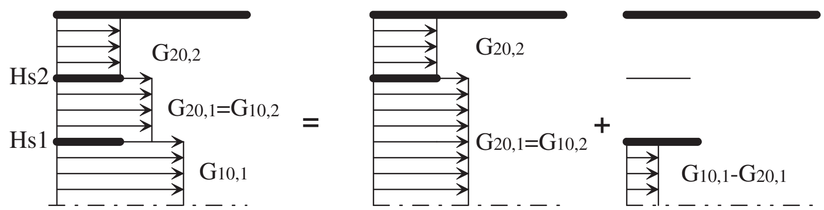

Figure 3. Decomposition of three-stream problem into (2) two-stream problems using superposition of conservative flux, e.g. momentum flux G.

Additionally, previous development efforts [3] have led to a multiple stream capability that is now fully functional. This method uses two-stream problems as the fundamental basis to build through superposition multiple stream problems. This concept is presented diagrammatically in Figure 3. As such, all the multiple stream and combustion capability is available within the scope of the periodic flow implementation.

As presented in Equations (1) and (2) the basic DREA formulation involves governing equations that may be written for the conservation quantities in terms of a general *linear* parabolic equation (the canonical form, e.g. Equation (2)). We emphasize the linear equations form because it is the linearity and the availability of superposition that permits the extension of the basic steady analysis in an efficient and general manner. Indeed it is the linearity in terms of conservative fluxes (and thereby superposition) of the governing equations that permits extension of the DREA equations to a far wider range of conditions than perhaps originally designed.

Recovery of primitive variable from conservative fluxes

For simplicity, we consider a single phase form of Equations (1) and (2) and note that it is written solely in terms of the conservative flux quantities:

$$\phi(x,y) \equiv \begin{pmatrix} \rho u^2 + p \\ \rho u H \\ \rho u \end{pmatrix} \tag{4}$$

To convert, these fluxes into primitive variables, such as (M, u, p, T, ρ), a local one-dimensional approximation is applied combined with the definitions of the fluxes themselves to compute the primitive variables [17]. Consider, for example, the velocity may be recovered from the flux values using:

$$\frac{\gamma+1}{2\gamma}[\rho u(x,y)]u^2 - [(\rho u^2 + p)(x,y)]u + \frac{\gamma-1}{\gamma}[\rho u H(x,y)] = 0 \tag{5}$$

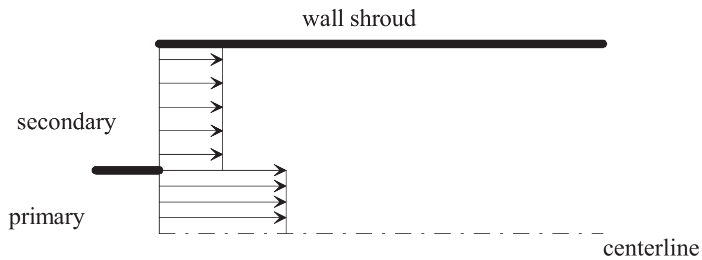

Figure 4. Schematic of flow initial conditions.

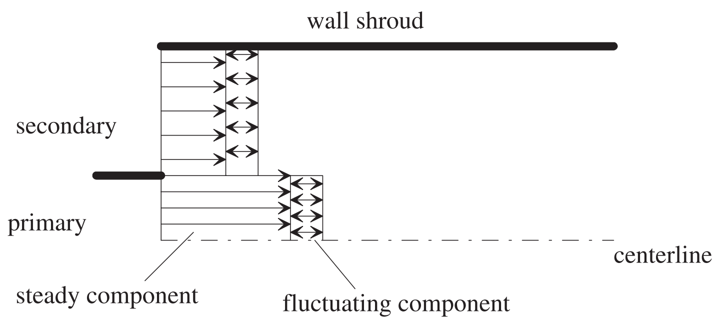

Figure 5. Periodic initial conditions for the mixing problem.

A somewhat more convenient form for analytical purposes of Equation (5) that is in terms of the conservative fluxes and the Mach number may be written:

$$\left(\gamma^2 - \frac{\gamma^2}{2} - C_0\right) M^4 + \left(2\gamma - \frac{\gamma^2}{\gamma - 1} C_0\right) M^2 + 1 = 0 \quad C_0 \equiv \frac{(\rho u^2 + p)^2}{(\rho u)(\rho u H)} \tag{6}$$

The two roots of the quadratic equation (in terms of M^2) correspond to supersonic and subsonic solutions for the ejector flow field. Hence, at every point in the flow field, the two basic ejector solutions are contained. One can also show that where the flow is choked, $M = 1$, implies a single solution. This corresponds to a zero discriminant in Equation (5). With this similarity in mind, it is apparent, that this equation has strong normal shock solutions embedded within it. This is consistent with our ejector analysis, which for negligible secondary flow must recover the classical-one-dimensional normal shock relationships.

Resolution of singular behaviour by analytical decomposition

The multiple stream mixing problems of interest are marked by singular behaviour near the splitter plate defining the initial conditions for these problems. Though an approximation to the physical problem, we can introduce a step function initial condition as a model for the actual initial condition. (see Figures 4 and 5).

As illustrated the flow is discontinuous at the interface between the primary and secondary streams. This rapid change will cause exceedingly poor performance for a strictly numerical integration method. Indeed we believe that one of the only possible ways to appropriately deal with this singularity is to introduce a local analytical solution that models the discontinuity. Examples of the use of special basis or trial function (Galerkin terminology) come from Fletcher [18]. In the problem considered here, rather than developing a local special differencing method and blending it back into the overall system, the linearity of the governing equation itself was used to perform a *global* decomposition. Fortunately a solution method from classical analysis, i.e. Green's functions, [19] is available which is based upon distribution theory rather than continuous functions and should provide a useable solution. The solution is written

$$\phi_{\mathrm{an}} = \frac{1}{2}(\phi_{10} - \phi_{20}) \sum_{-\infty}^{\infty} \left[\mathrm{erf}\left(\frac{y + h_{\mathrm{s}} - 2n}{(2a^*)^{1/2} x} \right) - \mathrm{erf}\left(\frac{y + h_{\mathrm{s}} - 2n}{(2a^*)^{1/2} x} \right) \right] + \phi_{20} \tag{7}$$

Note, that for $x \ll 1$ this relationship recovers the step input, i.e. Equation (7) (see References [1, 19] for further discussion). Although Equation (20) is exact, and does not suffer from the near field ($x \ll 1$) limitations that an eigenfunction expansion solution would, it is still in the form of an infinite series. However, in the near field, the solution converges very rapidly [1]. Using this rapid convergence what is implemented is a combined numerical and analytical solution of the problem. Since Equation (2) is linear, a composite analytical/numerical solution is easily effected using superposition. Note, that there is no matching or overlap region associated with the combined numerical analytical method described here, since both solutions are valid (and active) over the full solution domain. However, the contribution of the numerical solution near the singularity is very small but increases as the solution proceeds.

Numerical solution component

To obtain a high accuracy solution, implicit i.e. compact finite difference relationships are used to solve the 1-d parabolic partial differential equations. Differencing methods of this form have high accuracy in terms of truncation error, while requiring limited support [20, 21]. The streamwise marching portion of this problem is differenced using both Crank–Nicolson and a three-point backward fully implicit method. This type of formulation has been applied to a high efficiency, combined analytical/numerical fluid flow model. The implementation of these discrete numerical relationships, accuracy, grid-refinement and convergence considerations are described in detail in References [1, 4].

ANALYSIS

Basic periodic flow extension of the DREA conservation equations

Recalling that the basic DREA formulation involves governing equations that may be written for the conservation quantities in terms of the general *linear* parabolic equation, i.e. Equations (1) and (2)

$$\frac{\partial \phi}{\partial x} = a(x) \frac{\partial^2 \phi}{\partial y^2} = a^* x \frac{\partial^2 \phi}{\partial y^2}$$

it is trivial to write the periodic extension of these equations as

$$\frac{1}{U}\frac{\partial\tilde{\phi}}{\partial t}+\frac{\partial\tilde{\phi}}{\partial x}=a^{*}x\frac{\partial^{2}\tilde{\phi}}{\partial y^{2}} \tag{8}$$

where $\tilde{\phi}(x,y,t)=\mathrm{Re}(\mathrm{e}^{\mathrm{i}\omega t})\phi(x,y)=\cos\omega t\phi(x,y)$. The boundary and initial conditions must be of this form, i.e. sinusoidal in time.

The new initial conditions (chosen to construct a solution that honours the initial value described by Equation (9):

$$\tilde{\phi}(0,y,t)=\begin{cases}\phi_{10}\mathrm{e}^{\mathrm{i}\omega t}, & 0<y<h_{\mathrm{s}}\\ \phi_{20}\mathrm{e}^{\mathrm{i}\omega t}, & h_{\mathrm{s}}<y<H\end{cases} \tag{9}$$

Since we are restricted to periodic, unsteady flow, we do not have a temporal initial condition. Physically, we can model the unsteady periodic cycle of a system, e.g. an IC engine, but cannot model the actual 'start up' of an IC engine. Additionally, it is important to recognize the form of Equation (8) requires an approximate linearization that entails an associated level of modelling uncertainty. See Reference [5] for a discussion for the non-linear terms associated with reactive flows.

The unsteady equation is mapped back to a steady equation through the substitution $\tilde{\phi}(x,y,t)=\mathrm{Re}(\mathrm{e}^{\mathrm{i}\omega t})\phi(x,y)=\cos\omega t\phi(x,y)$ and simplification (see Reference [22]):

$$\mathrm{i}\frac{\omega}{U}\phi+\frac{\partial\phi}{\partial x}=a^{*}x\frac{\partial^{2}\phi}{\partial y^{2}} \tag{10}$$

Equations (9) and (10) outline the steady flow to periodic flow extension procedure. Though formally, a straightforward enhancement, implementation of periodic behaviour within the DREA framework is rather more complex. The major complexities of the extension involve:

- Extension of periodic results to simultaneously model steady base flow and periodic behaviour.
- Manipulation of multiple amplitude and frequency initial conditions, i.e. each stream contains its own periodic behaviour that mixes with other oscillating streams within the DREA framework.
- Solution of the modified steady equation, e.g. Equation (10). How is solution of the modified steady equation, especially since there may be considerable difficulty in solving boundary value problems in the presence of large (virtually singular) gradients, best performed?
- The previous analyses have been applied directly to scalar equations. What is the best structure for assigning periodic behaviour to conservation quantities and, through the decode step, to the primitive quantities themselves.

Periodic flow and base flow

Extension of periodic results to model both base flow and a fluctuating component is made essentially trivial by the linear nature of the DREA relationships. Indeed, using superposition valid for linear problems we can immediately propose a valid solution of the form:

$$\tilde{\phi}(x,y,t) = \mathrm{Re}[(1+\hat{a}(\mathrm{e}^{\mathrm{i}\omega t}))\phi(x,y)] = (1+\hat{a}\cos\omega t)\phi(x,y) \tag{11}$$

where $\hat{a}$ is a dimensionless, *constant* amplitude (estimated *a priori* or by other means to be a fraction of the base flow value) and be certain that it will satisfy the DREA conservation relationships. Note that both amplitude and frequency are dimensionless, i.e. $\hat{a} = \hat{a}_{\mathrm{dim}}/\phi_{\mathrm{ave.dim}}$, $\omega = \omega_{\mathrm{dim}}H/U$. Note that unlike many small disturbance linearization concepts, that due to the formal linearity of the DREA conservation relationships, $\hat{a}$ need not be necessarily small.

Solution of the modified steady equation in the presence of large gradients

Even the most elementary periodic flow transformation yields a steady state linear partial differential equation that must solved e.g. Equation (10). Although classical analytical techniques (Green's functions, eigenfunction expansion [19]) provide a straightforward solution to Equation (10), to stay within the current DREA construction, we originally considered the use of a fully numerical solution for this part of the problem.

However as alluded to previously, a strictly numerical integration of a problem like Equation (10) in the presence of (essentially) discontinuous initial conditions, e.g. Equation (4), gives very poor results due to the large gradients in the flow. Since we already have a combined numerical analytical method that works well for IVP's with discontinuity in the IC, modification of non-homogenous boundary value problems, BVP's form would be of great benefit. This transformation of a class of non-homogenous BVP (boundary value problem) to IVP's is referred to here as the *transformation solution method.* The development of the transformation is described in Appendix A.

Using the transformation solution method, solution of the equations like Equation (10) is simply:

$$\phi = \mathrm{e}^{-\mathrm{i}(\omega/U)x}\phi_{\mathrm{ss}}(x,y), \qquad \frac{\partial\phi_{\mathrm{ss}}}{\partial x} = a^{*}x\frac{\partial^2\phi_{\mathrm{ss}}}{\partial y^2} \tag{12}$$

By collecting terms and taking real parts we write:

$$\tilde{\phi}(x,y,t) = \mathrm{Re}\left[(1+\hat{a}(\mathrm{e}^{\mathrm{i}\omega t-\mathrm{i}(\omega/U)x}))\phi_{\mathrm{ss}}(x,y)\right] = \left[1+\hat{a}\cos\left(\omega\left(t-\frac{x}{U}\right)\right)\right]\phi_{\mathrm{ss}}(x,y) \tag{13}$$

Equation (13), provides a direct analytical connection between the system of IVP's in terms of $\phi_{\mathrm{ss}}(x,y$, i.e. the steady state solution which is already available using the combined analytical–numerical method and the periodic flow extension. It is worth noting, however, that the entire development to this point has been based upon a single amplitude and frequency. To assume that all streams, indeed the entire flow, resonates at the same frequency and amplitude would partially defeat the mixing capabilities of the DREA code. It is important that the formulation be generalized sufficiently such that each stream has its own individual periodic behaviour ($\hat{a}$ and ω). The discussion of this implementation is the subject of the next section.

Multiple streams with individual amplitude and frequency conditions

As shown previously the extension to period flow for a problem with a single period and amplitude is straightforward, one merely solves the steady equations for $\phi_{\mathrm{ss}}(x,y)$, applies the

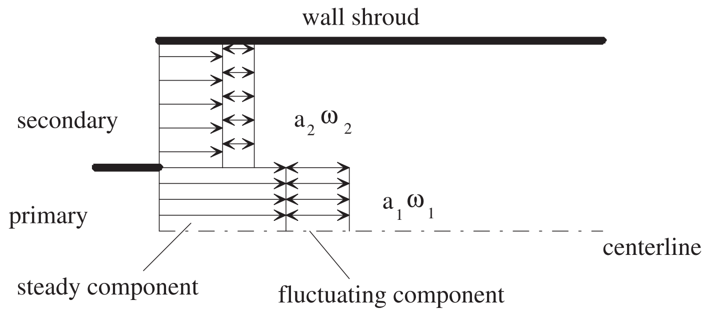

Figure 6. A two-stream periodic mixing problem with individual stream periodicities and amplitude.

transformation and appends the temporal result. Here we extend our analysis such that each stream has its own individual periodic behaviour ($\hat{a}$ and ω). Moreover, we will implement these ideas with the constraints of the currently available steady state DREA formulation.

Perhaps the best way to describe the multiple stream periodic flow problem is to consider a simple two-stream mix problem (shown in Figure 6) analytically using (say) an eigenfunction expansion solution for the elementary (but now periodically, unsteady) two-stream problem.

Referring to Figure 6 and recalling our prescription for extending steady base flow to a single ($\hat{a}$ and ω) periodic behaviour, i.e. solve the steady equations for $\phi_{ss}(x, y)$, apply the transformation and append the temporal result, the difficulty is obvious. We now have two streams, each with its own $\hat{a}$ and ω; how do we correctly 'append' the correction? Fortunately, we can fall back upon the linearity of the problem. We can write down (at least in an offline sense, since the DREA formulation will not easily admit this particular formulation) two single stream with a single amplitude and period problems, solve the associated problems and add the results. Figure 7 alludes to the necessary decomposition.

This decomposition, however, is at a more fundamental level than the DREA code, which uses the two-stream problem as its fundamental problem, but it is useful to show how periodic flows can be completely resolved. The associated analysis for the eigenfunction expansion solutions is straightforward [19] and without going through the solution process we merely quote the result. The primary stream problem gives:

$$\tilde{\phi}_1(x, y, t) = \bar{\phi}_1 + \sum_{n=1}^{\infty} a_{n,1} \cos\left(\frac{n\pi y}{H}\right) e^{-a^*(n\pi/H)^2 x^2/2}$$

$$a_{n,1} = -\frac{2\phi_{10}}{n\pi} \sin(n\pi h_s) \left[1 + \hat{a}_1 \cos\left(\omega_1 \left(\frac{x}{U} - t\right)\right)\right] \tag{14}$$

$$\bar{\phi}_1 = \phi_{10} h_s \left[1 + \hat{a}_1 \cos\left(\omega_1 \left(\frac{x}{U} - t\right)\right)\right]$$

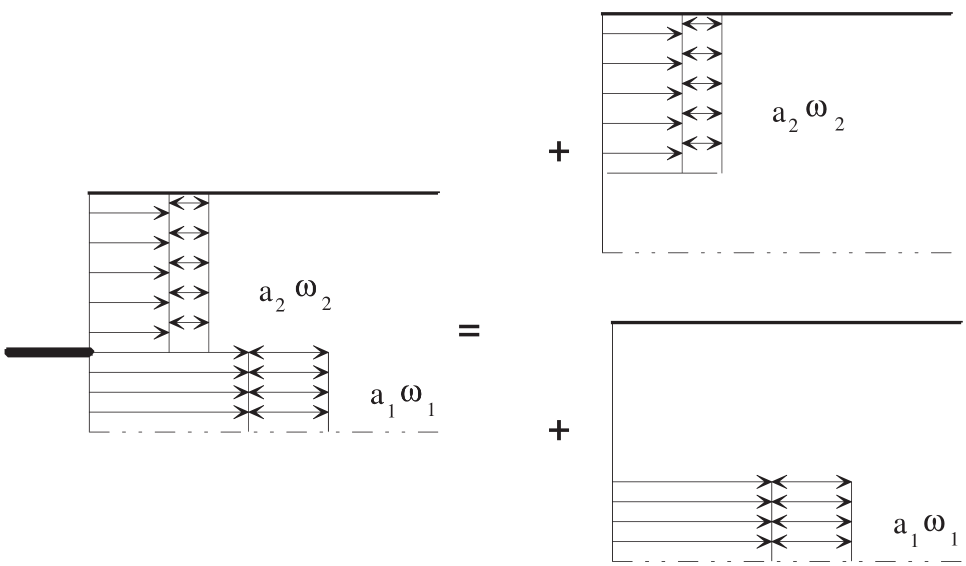

Figure 7. Decomposition of the two-stream problem periodic problem into two single stream problems.

while the secondary stream gives

$$\tilde{\phi}_2(x,y,t) = \bar{\phi}_2 + \sum_{n=1}^{\infty} a_{n,2}\cos\left(\frac{n\pi y}{H}\right) \mathrm{e}^{-a^*(n\pi/H)^2x^2/2}$$

$$a_{n,2} = -\frac{2\phi_{20}}{n\pi}\sin(n\pi h_{\mathrm{s}})\left[1+\hat{a}_2\cos\left(\omega_2\left(\frac{x}{U}-t\right)\right)\right] \tag{15}$$

$$\bar{\phi}_2 = \phi_{20}(1-h_{\mathrm{s}})\left[1+\hat{a}_2\cos\left(\omega_2\left(\frac{x}{U}-t\right)\right)\right]$$

The total solution is merely the sum of Equations (14) and (15). Thus, at least formally, one can write a solution that honours the multiple stream/multiple periodic flow problem. It is also worth noting, that the solution space is now proceeding along kinematic wave characteristics, i.e. MOC of the form: $s=t-x/U$.

Unfortunately, building multiple stream solutions using single stream superposition is not the structure of the DREA code. (The DREA code uses two-stream problems as the basis of its superposition method [3]). Multiple stream problems for $N>2$ (N is the number initial condition streams) are also built on 2-stream basic problems, recall Figure 3, where $N=3$. Hence, it is not possible to (easily) use the current DREA formulation or coding to build multiple stream periodic flow problems.

However, if one carefully examines the eigenfunction superposition solution, i.e. Equation (14) plus Equation (15), one notes that the periodicity modifications, e.g. the term $[1+\hat{a}_i\cos(\omega_i(t-x/U))]$ is directly aligned with the particular scalar for that stream, such as ϕ_{10} or ϕ_{20}. This can be seen by inspection of Equation (16) which is the sum of Equations (14) and (15):

$$\tilde{\phi}(x,y,t)=\bar{\phi}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi y}{H}\right)\mathrm{e}^{-a^*(n\pi/H)^2x^2/2}$$

$$a_n=\frac{2}{n\pi}\left(\phi_{20}\left[1+\hat{a}_2\cos\left(\omega_2\left(\frac{x}{U}-t\right)\right)\right]\right.$$
$$\left.-\phi_{10}\left[1+\hat{a}_1\cos\left(\omega_1\left(\frac{x}{U}-t\right)\right)\right]\right)\sin(n\pi h_\mathrm{s}) \tag{16}$$
$$\bar{\phi}=\phi_{10}\left[1+\hat{a}_1\cos\left(\omega_1\left(\frac{x}{U}-t\right)\right)\right]h_\mathrm{s}$$
$$+\phi_{20}\left[1+\hat{a}_2\cos\left(\omega_2\left(\frac{x}{U}-t\right)\right)\right](1-h_\mathrm{s})$$

Further, the positioning of the appropriate periodic flow modification terms Equation (16) provides a motivation behind an effective substitution that utilizes the analytical portion of the combined analytical/numerical DREA method. If one introduces the definitions for an ‘effective’ initial condition:

$$\begin{aligned}\phi_{10,\mathrm{eff}}&=\phi_{10}\left[1+\hat{a}_1\cos\left(\omega_1\left(\frac{x}{U}-t\right)\right)\right]\\ \phi_{20,\mathrm{eff}}&=\phi_{20}\left[1+\hat{a}_2\cos\left(\omega_2\left(\frac{x}{U}-t\right)\right)\right]\end{aligned} \tag{17}$$

then, in terms of these variables that the same steady state problem is recovered.

$$\begin{aligned}\phi(x,y)&=\bar{\phi}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi y}{H}\right)\mathrm{e}^{-a^*(n\pi/H)^2x^2/2}\\ a_n&=\frac{2}{n\pi}(\phi_{20,\mathrm{eff}}-\phi_{10,\mathrm{eff}})\sin(n\pi h_\mathrm{s})\\ \bar{\phi}&=\phi_{10,\mathrm{eff}}h_\mathrm{s}+\phi_{20,\mathrm{eff}}(1-h_\mathrm{s})\end{aligned} \tag{18}$$

This close correspondence of the periodic solution in terms of the effective initial conditions with the steady state solution provides a modification to the DREA framework that will permit multiple stream periodic flow behaviour; namely:

- Introduce effective, i.e. Equation (17), initial conditions, ϕ_eff.
- Effect a steady DREA solution. Note that since these effective initial conditions are not constants, they are actually functions of x and t, it is not possible to effectively use the numerical integration portion of the analytical–numerical DREA method (if one were to attempt these integrations, it would be necessary to ‘march’ to each new ‘x’ location with a new initial condition for each step, a very inefficient proposition. Although the DREA model does not actually use an eigenfunction expansion based analytical solution, it uses Green’s functions and the method of images e.g. Equation (7), the prescription remains completely unchanged. Because the Green’s function expansion is a rapidly convergent series, one finds that $n>2$ yields an invariant solution. For example, a subsonic, two-stream mixer with $M_1=0.22$, $M_2=0.01$, gives a variation in the wall velocity, $y/H=1$

(the location where convergence of the series is most difficult) gives 3 digit accuracy for $n=1$ and 5 digits for $n=2$. Supersonic and transonic problems yield similar convergence rates.

- Note that periodic flow algorithm relies solely on the analytical solver, as such, the accuracy of the computation is independent of the grid and grid spacing. Indeed, the grid represents simply an evaluation location rather than a portion of the solution.

The straightforward modification that we have presented permits us to use the current DREA implementation while extending the solution to model periodic flow. Next we discuss possible (and as we shall see non-unique) 'inversion' methods to obtain primitive variables.

Assignment and decode of conservative and primitive period variables for periodic flows

The previous analyses have been applied directly to single scalar equations (always in terms of a conservation quantity e.g. ϕ, which represents conservative fluxes such as ρu, ρu^2+p, ρuH, etc.). As such, it has not been necessary to differentiate between assignment of periodic behaviour to the conservative quantities and the associated primitive variable. Recalling from Equations (5) and (6) that these conservative quantities are 'decoded' or solved for the associated primitive quantities, such as, density, temperature velocity, etc. In this section we discuss possible 'best' structures for assigning periodic behaviour to conservation quantities and, through the decode step, to the primitive quantities themselves.

To extend the conservative quantities to periodic flow, we must assign periodic behaviour to the conservative terms, i.e. the diagonal terms in Equation (19):

$$\phi(x,y,t)=\begin{pmatrix}(1+\hat{a}\cos\omega_G t) & 0 & 0 & 0\\ 0 & (1+\hat{a}\cos\omega_{\rho uH}t) & 0 & 0\\ 0 & 0 & (1+\hat{a}\cos\omega_{\rho uH}t) & 0\\ 0 & 0 & 0 & (1+\hat{a}\cos\omega_{\rho uY_i}t)\end{pmatrix}\times\begin{pmatrix}\rho u^2+p\\ \rho uH\\ \rho u\\ \rho uY_i\end{pmatrix} \quad (19)$$

One obvious choice for the periodic terms in Equation (19) is to demand equality of periodic terms, i.e. $(1+\hat{a}\cos\omega_G t)=(1+\hat{a}\cos\omega_{\rho uH}t)=(1+\hat{a}\cos\omega_{\rho u}t)=(1+\hat{a}\cos\omega_{\rho uY_i}t)$. This choice, however, is not well posed when we attempt to use the inversion methods to compute the Mach number through Equation (6) since

$$C_0=\frac{(G(1+\hat{a}\cos\omega_G t))^2}{((\rho u)(1+\hat{a}\cos\omega_{\rho uH}t)(\rho uH)(1+\hat{a}\cos\omega_{\rho u}t))}=\frac{(G_{ss})^2}{((\rho u_{ss})(\rho uH_{ss}))} \quad (20)$$

Clearly with equality of periodic flow, all dimensionless or even scaled quantities, e.g. $\rho uH/\rho u$, etc. will lose their dependence on periodic flow, which we deem unacceptable.

A physically based (though still not unique) assignment of periodic behaviour treats the various primitive/non-conservative quantities such as, u, p, T, ρ and Y_i with a set periodic value. If one chooses ρ, u, Y_i as fundamental quantities so that

$$\begin{pmatrix} \rho(x,y,t) \\ u(x,y,t) \\ Y_i(x,y,t) \end{pmatrix} = \begin{pmatrix} (1+\hat{a}\cos(\omega t))\rho(x,y) \\ (1+\hat{a}\cos(\omega t))u(x,y) \\ (1+\hat{a}\cos(\omega t))Y_i(x,y) \end{pmatrix} = \begin{pmatrix} (1+\hat{a}\ \mathrm{Re}(\mathrm{e}^{\mathrm{i}\omega t}t))\rho(x,y) \\ (1+\hat{a}\ \mathrm{Re}(\mathrm{e}^{\mathrm{i}\omega t}t))u(x,y) \\ (1+\hat{a}\ \mathrm{Re}(\mathrm{e}^{\mathrm{i}\omega t}t))Y_i(x,y) \end{pmatrix} \tag{21}$$

Then from the definitions of the conservative quantities, i.e. G, ρuH, ρu, ρuY_i it is possible immediately to obtain appropriate values:

$$\begin{pmatrix} G(x,y,t) \\ \rho u(x,y,t) \\ \rho uH(x,y,t) \\ \rho uY_i(x,y,t) \end{pmatrix} = \begin{pmatrix} (1+\hat{a}\cos(3\omega t))(\rho u^2+p) \\ (1+\hat{a}\cos(2\omega t))p(x,y) \\ (1+\hat{a}\cos(4\omega t))u(x,y) \\ (1+\hat{a}\cos(3\omega t))Y_i(x,y) \end{pmatrix} \tag{22}$$

As a corollary to the expressions in Equation (22) we also require that $p(x,y,t) = (1+\hat{a}\cos(3\omega t))$ while $T(x,y,t) = (1+\hat{a}\cos(2\omega t))$ since we must be able to factor periodic behaviour from these terms.

With these definitions, the periodic conditions are completely specified. Inversion is precisely as before, except it is no longer necessary to be concerned about the loss of periodic behaviour.‡ Thus the periodic problem is now fully specified: given the frequency, ω and amplitude $\hat{a}$ for each stream, it is now possible to compute the effect of periodicity within the mixing flow. As such, several examples and parametric problems that demonstrate these effects are investigated.

RESULTS

Here, we discuss results associated with the periodic problem. Validity of the basic steady DREA implementation is demonstrated in References [1, 2, 4] by comparison with several simple, single fluid, non-reactive (cold flow), steady ejector–nozzle experiments [23, 24]. Reference [5] considers reactive flow mixing problems.

‡Note that it would appear that one again seems to lose periodic effects if one considers

$$\frac{(G\mathrm{e}^{\mathrm{i}3\omega t})^2}{((\rho u\mathrm{e}^{\mathrm{i}2\omega t})(\rho uH\mathrm{e}^{\mathrm{i}4\omega t}))} = \frac{(G)^2}{((\rho u)(\rho uH))}$$

but we are interested in applying real components within this relationship, i.e.

$$\frac{\mathrm{Re}(G\mathrm{e}^{\mathrm{i}3\omega t})^2}{(\mathrm{Re}(\rho u\mathrm{e}^{\mathrm{i}2\omega t})\mathrm{Re}(\rho uH\mathrm{e}^{\mathrm{i}4\omega t}))} \neq \frac{(G)^2}{((\rho u)(\rho uH))}.$$

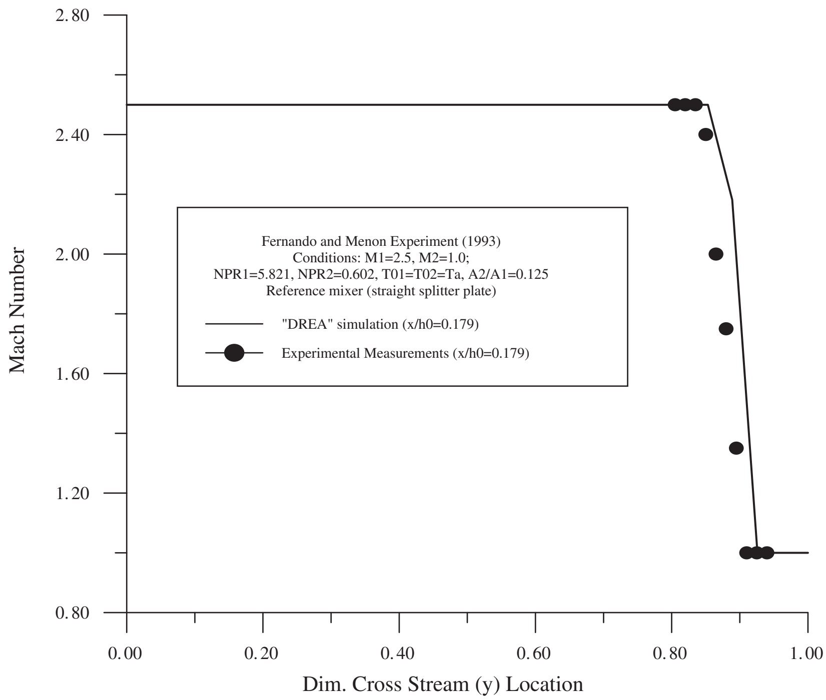

Figure 8. A comparison between the DREA simulation and experimental data of [24] showing the Mach number profile for a slot mixer.

Consider for example second problem is a steady supersonic (Mach 2.5) and, choked flow (Mach 1.0) mixing problem studied by [24], Figure 8. This study involves a 2-d mixing layer developed by a 2-d slot ejector. Comparison with experiment is good, however, the physical location of the shear layer is not exactly predicted. It is suspected, that wave expansion/compression effects around the step are modifying the interface slipline. Rate of spreading of the mixing layer, which is closely related to the turbulence models used in the DREA code is discussed in detail in References [2, 25].

Now, concentrating on the focus of this research, i.e. periodic flow problems, we develop a series of parametric and code validation problems. Unfortunately direct experimental results are limited, and we will concentrate on parametric/demonstrative computations to delineate the limitations and capabilities of the periodic flow modifications. However, the use of periodic excitation to control (typically enhance) mixing has been the subject of a number of studies, e.g. [26, 27]. A detailed study that provides mixing data for subsonic, axi-symmetric jet problems with periodic excitation is discussed by Zaman and Raman [28]. Figure 9 depicts the centerline velocity versus streamwise distance for flow with and without excitation.

As shown in Figure 9, the overall trend of increased mixing due to excitation is properly predicted by the DREA model. Agreement between experiment and the DREA simulation is moderate for both flow with and without excitation. Since DREA is a 2-d simulation Cartesian rather than axi-symmetric, this level of agreement is considered acceptable.

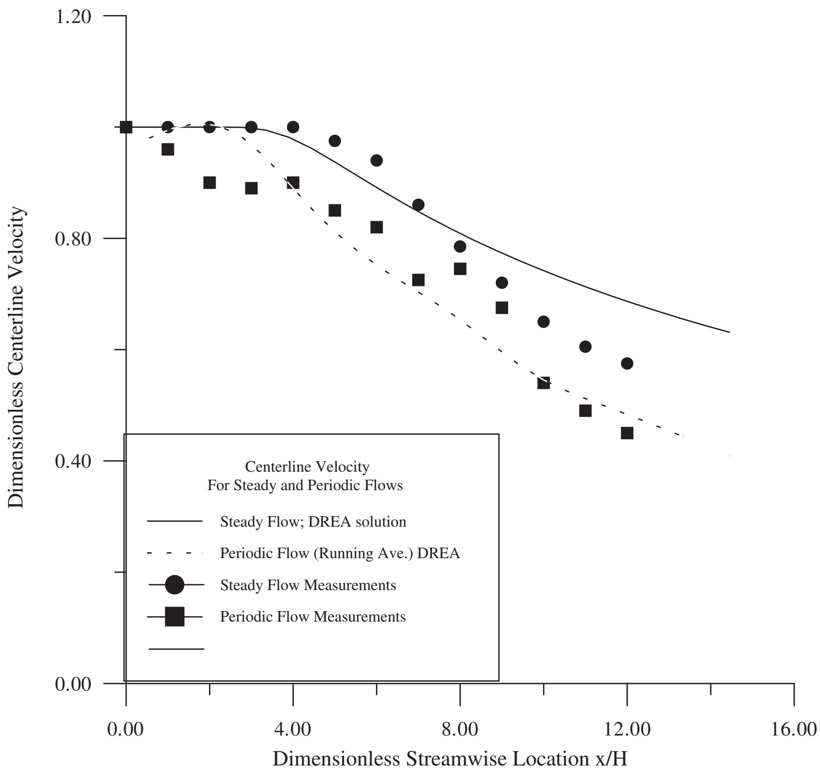

Figure 9. Comparison between experimental measurement [28] and DREA simulation dimensionless centerline velocity for flow with periodic excitation; i.e. $\omega = 16.8$ and $\hat{a} = 0.025$ with $t = 1.0$. Flow without excitation is presented as well for comparison.

To gain a sense of the instantaneous flow field, we consider a simple two-stream periodic flow problem as a starting point. Both the primary stream and secondary stream have the same period and amplitude, i.e. $\omega = 0.5$ and $\hat{a} = 0.025$ in Figure 10.

As a useful comparison of the two-stream periodic problem we compare to the same two-stream steady-state problem in Figure 11.

A simple demonstration of mixing is where the primary stream has period and amplitude, i.e. $\omega_1 = 0.5$ and $\hat{a}_1 = 0.025$, while the secondary is steady, i.e. $\omega_2 = 0$, and $\hat{a}_2 = 0.0$ is presented in Figure 12. Note that this flow is analogous with the jet problem, i.e. Figure 8, except the primary stream area is much smaller than the secondary stream area and the secondary stream is quiescent.

The periodic extension analysis also includes multiple stream effects and reactive flow processes. Consider the simple multiple (three stream problem) non-reactive flow problem presented in Figure 13:

Finally, we note, the periodic flow extension is valid for combustion flows as well. As an appropriate example, consider a two-stream, SCRAMJET problem with hydrogen burn analogous to the model discussed by Oevermann [29]. The Mach field for this problem is presented in Figure 14.

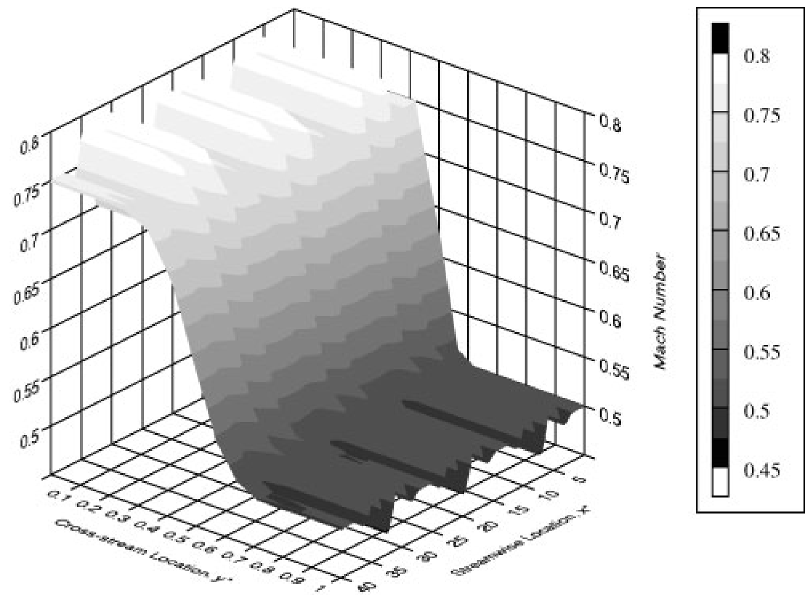

Figure 10. Two-stream subsonic, $M_1 = 0.75$, $M_2 = 0.5$ periodic flow problem Mach numbers. Both the primary stream and secondary stream have the same period and amplitude, i.e. $\omega = 0.5$ and $\hat{a} = 0.025$ with $t = 1.0$.

CONCLUSIONS

Here we have presented an extension from strictly steady flow to periodic flow for the aerodynamic mixing code DREA (differential reduced ejector analysis). Using the fact that reduction of linear (or linearizable) periodic governing to steady state is a classical problem, we discussed these forms as motivation to our implementation. However, we concluded that, although simple relationships are easily extended to period flow, extension of systems of equations and (moreover) problems with complex initial conditions are challenging to extend. The inherent large gradient initial condition singularities (which have greatly influenced all portions of the DREA code development) placed considerable limitations on the use of numerical solution methods. Fortunately, however, the unique combined analytical–numerical form of the DREA formulation provided a successful solution method. Comparison with experimental measurements for jet flows with excitation show moderate to good agreement with the simulation. Other flow fields are presented to demonstrate the capabilities of the model. Finally we emphasize that we have retained through this process the simple, efficient, extremely coarse grid DREA structure that has been the original intent of the DREA development effort. The simplicity and efficiency of the DREA code continue to make it uniquely suitable for its original niche, namely design and preliminary design environments where more complex and expensive models are inappropriate.

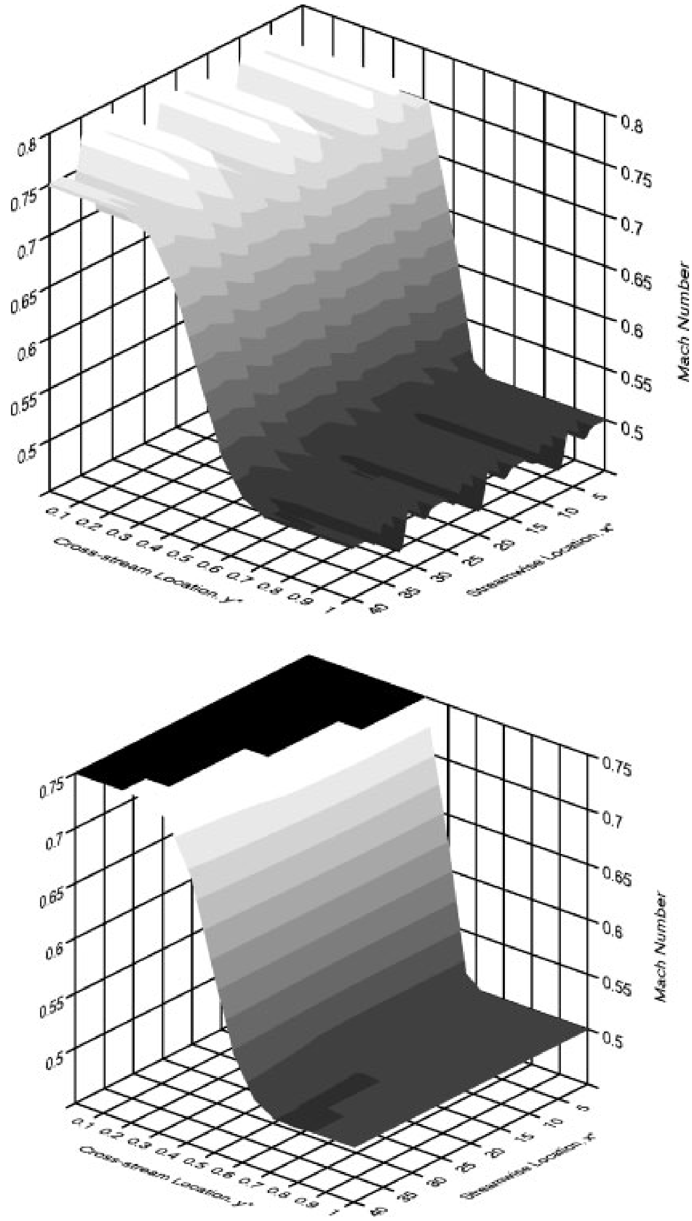

Figure 11. Comparison between the Mach number fields for two-stream subsonic, periodic problem $M_1 = 0.75$, $M_2 = 0.5$, $\omega = 0.5$ and $\hat{a}_1 = 0.025$, with $t = 1.0$ with the two-stream steady problem, $M_1 = 0.75$, $M_2 = 0.5$.

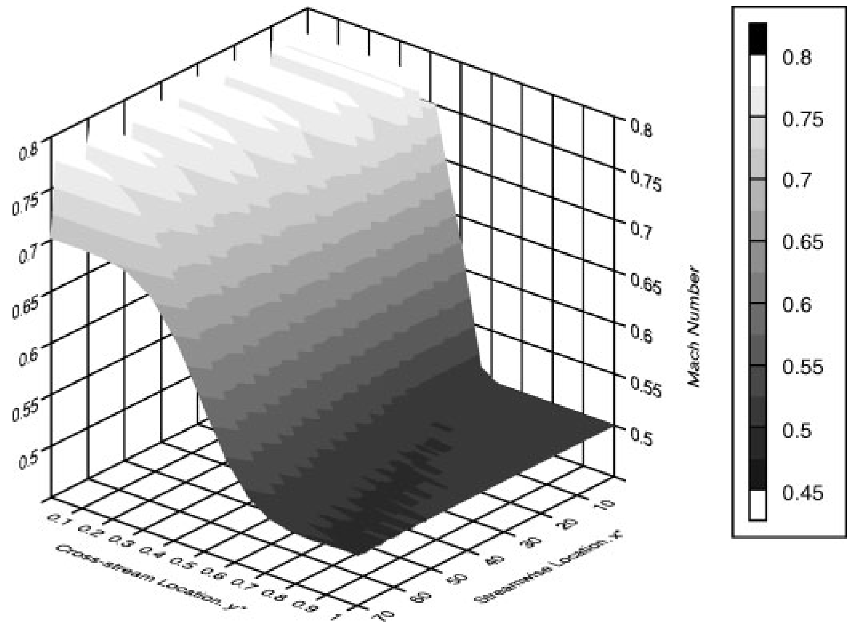

Figure 12. Mach number field for two-stream subsonic $M_1 = 0.75$, $M_2 = 0.5$ problem. Primary stream has period and amplitude, i.e. $\omega_1 = 0.5$ and $\hat{a}_1 = 0.025$, while the secondary is steady, i.e. $\omega_2 = 0.0$ and $\hat{a}_2 = 0.0$ Notice the growth of periodic behaviour in the secondary stream due to mixing.

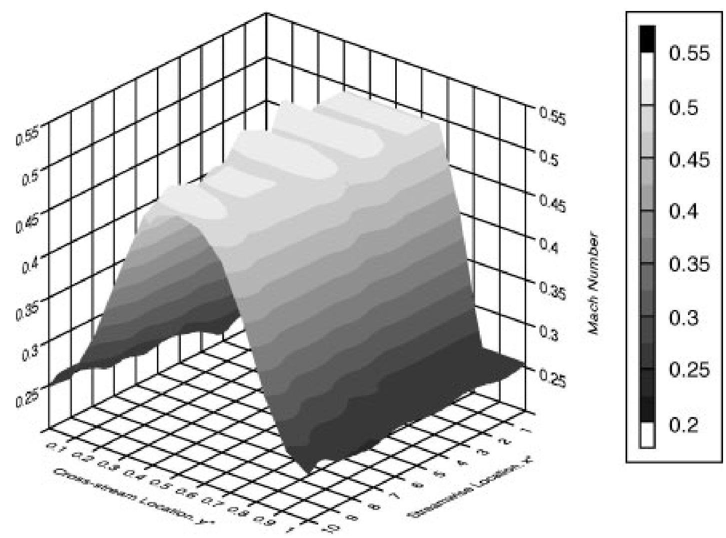

Figure 13. Mach number field for three stream periodic mixing problem, $M_1 = M_3 = 0.25$ and $M_2 = 0.5$. All amplitudes $= 0.025$, and $\omega_1 = \omega_3 = 0.1$ while $\omega_2 = 1$.

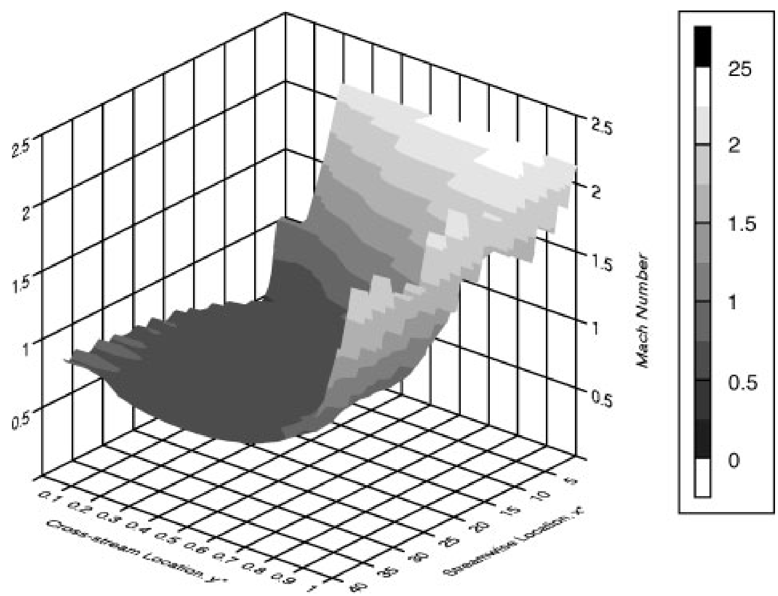

Figure 14. Mach number field for two-stream periodic mixing/combustion problem with $M_1 = 1.0$, $M_2 = 2.0$. The associated amplitudes $\hat{a}_1 = 0.025$, $\hat{a}_2 = 0.025$ and periodicities $\omega_1 = 1$, $\omega_2 = 0.5$.

APPENDIX A: TRANSFORMATION SOLUTION METHOD

As alluded to previously, a strictly numerical solution of the source term problems yields very poor results due to the large gradients in the flow. Indeed these results are as bad as trying to solve the IC problem (also called an IVP, initial value problem) strictly numerically. As such, the proposed, simple-minded splitting cannot be successful, while a more mathematically substantial method is. The previously described coupled system solution method is just such a technique. Here we briefly describe the generic problem with large gradient source terms and the analogy with discontinuous initial condition problems. This will lead us to consider a solution method to model large gradient source term problems.

Consider the coupled inhomogeneous boundary value problem:

$$\frac{\partial u}{\partial x} = a(x)\frac{\partial^2 u}{\partial y^2} + S(u, v)$$

$$\frac{\partial v}{\partial x} = a(x)\frac{\partial^2 v}{\partial y^2} \tag{A.1}$$

with the initial condition:

$$u(0, y) = 0$$

$$v(0,y) = \begin{pmatrix} v_{10} 0 \leqslant y \leqslant h_s \\ v_{20} h_s < y \leqslant 1 \end{pmatrix} \quad \text{(A.2)}$$

The close resemblance to the coupled system problem is obvious, except $S(u,v)$ is not permitted to be a second derivative like v_{yy} which make the use of the coupled solution method impossible. For simplicity let $S(u,v)=v$. Our naïve solution method was to compute (using the accurate IC solver) $v(x,y)$ and substitute into the 'u' equation. However even for this simple problem, one can see this will fail... $v(x,y)$ is virtually discontinuous for $x \leqslant 1$ because it honors the IC in Equation (A.1). Thus, $S(u,v)=v$ will also be virtually discontinuous and if we could not solve the IC problem numerically, the inhomogeneous source term will be no better.

Indeed, the numerical solution difficulties that one faces for IC problems are similar to the difficulty in numerically solving for inhomogeneous source term problems. But this gives us a strategy to solve source problems. Since the problems are similar, perhaps we can modify some source term problems into equivalent IC problems. A modification of this form would be of benefit, since we already have powerful IVP (initial value problem) solvers. This transformation of special (but useful for us) inhomogenous BVP (boundary value problem) to IVPs is referred to here as the *transformation solution method.*

Consider the simple IVP with the added linear term 'u':

$$\frac{\partial u}{\partial x} = a(x)\frac{\partial^2 u}{\partial y^2} + u \quad \text{(A.3)}$$

and the IC (it would yield the trivial solution otherwise):

$$u(0,y) = \begin{pmatrix} u_{10} 0 \leqslant y \leqslant h_s \\ u_{20} h_s < y \leqslant 1 \end{pmatrix} \quad \text{(A.4)}$$

Can Equation (A.2) be placed into the simple heat equation form? The answer is yes. Consider the substitution $u=v(x,y)w(x)$.

$$\frac{dw}{dx}v + w\frac{\partial v}{\partial x} = a(x)w\frac{\partial^2 v}{\partial y^2} + vw \quad \text{(A.5)}$$

Now splitting the differential equation into two GDEs:

$$\frac{\partial v}{\partial x} = a(x)\frac{\partial^2 v}{\partial y^2}$$

$$\frac{dw}{dx} = w \Rightarrow w = e^x \quad \text{(A.6)}$$

Equation (A.6) represents a new transformed system IC problem where we have put the non-standard form of Equation (A.1) back into standard form (a simple heat equation) using the new variable $u = e^x v(x, y)$. Notice as well that the IC for the $v(x, y)$ remains unchanged since $u(0, y) = e^0 v(0, y)$.

APPENDIX B: NOMENCLATURE

$\hat{a}$	periodic amplitude
$a(x)$	turbulence model
a^*	turbulence constant
f	canonical scalar function
G	momentum flux, $\rho u^2 + p$
G_{mod}	modified momentum flux, $\rho u^2 + \omega p$
k	reaction rate
h	enthalpy, height
H	total enthalpy, total channel height
L	streamwise length scale
M	Mach number
Pr	Prandtl number
p	pressure
R	ideal gas constant
Re	real part of complex number
S	canonical source term
Sc	Schmidt number
T	temperature
t	time
U	average velocity
u	streamwise velocity, canonical coupled scalar
v	cross-stream velocity, canonical coupled scalar, transformation variable
w	generation term, transformation variable
x	dimensionless streamwise spatial co-ordinate, x/H
Y	mass fraction
y	dimensionless cross-stream spatial co-ordinate, y/H

Greek letters

α	coupled problem coefficient
δ	mixing layer thickness
γ	ratio of specific heats
ν	kinematic viscosity, stoichiometric coefficient
ω	Vigneron parameter, periodic frequency
ϕ	scalar flux
$\tilde{\phi}$	unsteady periodic scalar flux
ρ	density

Subscripts

0	constant value
ave	weighted average, e.g. $\phi_{10}h_s + \phi(1 - h_s)$
eff	effective value
f	fuel
i	streamwise co-ordinate, species
j	cross-stream co-ordinate
o	oxidizer
p	product
s	splitter plate
ss	steady state
t	turbulent
10	'primary' stream initial condition
20	'secondary' stream initial condition

Superscripts

IC	initial condition problem
'	turbulent fluctuation quantity

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REFERENCES

1. De Chant LJ, Seidel JA. A combined analytical singularity subtraction method and compact implicit differencing scheme for aerodynamic mixing. *Mathematical and Computer Modelling* 2000; **31**:21–38.
2. De Chant LJ, Seidel JA. Simplified turbulence modelling for forced vortical mixers using a kinematic wave analogy. *Applied Mathematical Modelling* 2001; **25**:427–447.
3. De Chant LJ, Seidel JA. Multiple stream superposition of a two-stream combined analytical numerical initial value problem solver for aerodynamic mixing. *Computers and Mathematics with Applications* 2002; **43**: 1451–1468.
4. De Chant LJ. Combined numerical/analytical perturbation solutions of the Navier–Stokes equations for aerodynamic ejector/mixer nozzle flows. NASA Contractor Report 1998-207406, 1998.
5. De Chant LJ, Seidel JA, Kline TR. Extension of a combined analytical/numerical initial value problem solver for turbulent mixing with combustion. *Computers and Fluids*, 04-02, submitted.
6. Drazin PG, Reid WH. *Hydrodynamic Stability*. Cambridge University Press: London, 1981.
7. Dutton JC, Carroll BF. Optimal supersonic ejector designs. *Transactions of the ASME*, *Journal of Fluids Engineering* 1986; **108**:414–420.
8. Dutton JC, Carroll BF. Limitation of ejector performance due to exit choking. *Transactions of the ASME, Journal of Fluids Engineering* 1988; **110**:91–93.
9. Ginoux JJ (ed.). Supersonic ejectors. AGARDograph 163, AGARD-AG-163, 1973.
10. Addy AL, Dutton JC, Mikkelsen SCDT. Supersonic ejector–diffuser theory and experiments. Report UILU ENG-82-4001, Department of Mechanical and Industrial Engineering, University of Illinois at Urbana-Champaign, Urbana, IL, 1982.
11. Fabri J, Siestrunck R. Supersonic air ejectors. In *Advances in Applied Mechanics*, vol. V. Academic Press: New York, 1958; 1–34.
12. Chow WL, Yeh PS. Characteristics of supersonic ejector systems with non-constant area shroud. *AIAA Journal* 1965; **3**:526–527.

13. Hedges KR, Hill PG. Compressible flow ejectors. Part I: development of a finite-difference flow model. *Transactions of the ASME*, *Journal of Fluids Engineering* 1974; **96**:272–281.
14. Hedges KR, Hill PG. Compressible flow ejectors. Part II: flow field measurements and analysis. *Transactions of the ASME*, *Journal of Fluids Engineering* 1974; **96**:282–288.
15. Williams FA. *Combustion Theory*. Addison-Wesley: Reading MA, 1985.
16. White FM. *Viscous Fluid Flow* (2nd edn). McGraw-Hill: New York, 1991.
17. Anderson DA, Tannehill JC, Pletcher RH. *Computational Fluid Mechanics and Heat Transfer*. Hemisphere: New York, 1984.
18. Fletcher CAJ. *Computational Galerkin Methods*. Springer-Verlag: New York, 1984.
19. Haberman R. *Elementary Applied Partial Differential Equations*. Prentice-Hall: Englewood Cliffs, NJ, 1983.
20. Hirsch C. *Numerical Computation of Internal and External Flows*: *vols* 1 *and* 2. Wiley: New York, 1988.
21. Hirsch C. Higher order accurate difference solutions of fluid mechanics problems by a compact difference scheme. *Journal of Computational Physics* 1975; **19**:90–109.
22. De Chant LJ. A perturbation model for the oscillatory flow of a Bingham plastic in rigid and elastic tubes. *Transactions of the ASME*, *Journal of Biomechanical Engineering* 1999; **121**:502–504.
23. Gilbert GB, Hill PG. Analysis and testing of two-dimensional slot nozzle ejectors with variable area mixing sections. *NASA Contractor Report* CR-2251, 1973.
24. Fernando EM, Menon S. Mixing enhancement in compressible mixing layers: an experimental study. *AIAA Journal* 1993; **31**(2):278–285.
25. De Chant LJ, Andrews MJ. Growth rates for compressible turbulent mixing layers using a nonlinear stability method. AIAA 9th International Space Plane and Hypersonic System and Technologies Conference, Norfolk, VA 11-01-99, AIAA-99-4861.
26. Zaman KBM, Hussain KMF. Vortex pairing in a circular jet under controlled excitation. Part 1. General jet response. *Journal of Fluid Mechanics* 1980; **101**:449.
27. Raman G, Rice EJ. Axi-symmetric jet forced by fundamental and subharmonic tones. *AIAA Journal* 1991; **29**:1114.
28. Zaman KBM, Raman G. Reversal in spreading of a tabbed circular jet under controlled excitation. *Physics of Fluids* 1997; **9**(12):3733.
29. Oevermann M. Numerical investigation of turbulent hydrogen combustion in a SCRAMJET using flamelet modelling. *Aerospace Science and Technology* 2000; **4**:463–480.

